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# SUMMATION OF A DOUBLE SERIES.*

BY T. H. GRONWALL.

It is the purpose of the present note to show that the series

$$(1) \qquad F(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(m+n-2)!\,(m+n-1)!}{m!\,(m-1)!\,n!\,(n-1)!} x^{2m}y^{2n},$$

which occurs in a physical problem,† has its region of convergence defined by $|x| + |y| < 1$, and that its sum is

$$(2) \qquad F(x, y) = \frac{1}{2}[1 - x^2 - y^2 \\ - \sqrt{(1 + x + y)(1 + x - y)(1 - x + y)(1 - x - y)}],$$

where that branch of the square root is to be taken which reduces to $+ 1$ at $x = y = 0$.

We begin by showing that our series converges absolutely and uniformly for $|x| + |y| \leqq 1 - \epsilon$, where $\epsilon$ is as small as we please. The binomial expansion of $(|x| + |y|)^{m+n}$ contains only non-negative terms, one of which is $\frac{(m+n)!}{m!\,n!} |x|^m |y|^n$; consequently

$$\frac{(m+n)!}{m!\,n!} |x|^m |y|^n \leqq (1 - \epsilon)^{m+n},$$

and therefore also

$$\frac{(m+n-2)!}{(m-1)!\,(n-1)!} \cdot \frac{(m+n-1)!}{m!\,n!} |x|^{2m}|y|^{2n} \leqq \frac{1}{n} |x|\,|y|^2(1 - \epsilon)^{2m+2n-3} \\ \leqq (1 - \epsilon)^{2m+2n}.$$

The series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1 - \epsilon)^{2m+2n}$ being obviously convergent, our statement is proved.

By a well-known theorem on power series, (1) may be differentiated term by term, and we obtain

$$(3) \qquad \frac{1}{2y}\frac{\partial F(x, y)}{\partial y} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(m+n-2)!\,(m+n-1)!}{m!\,(m-1)!\,(n-1)!\,(n-1)!} x^{2m}y^{2n-2}$$

for $|x| + |y| < 1$.

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* Read before the American Mathematical Society, Feb. 26, 1921.

† K. W. Lamson, "Reflection of radiation from an infinite series of equally spaced planes," Physical Review, ser. 2, vol. 17 (1921), pp. 624–625.

Now assume that

$$(4) \qquad |x| < \frac{1}{2\sqrt{2}}, \qquad |y| < \frac{1}{2\sqrt{2}};$$

then $|x| + |y| < 1$, so that (3) converges absolutely, and for every $z$ on the circle $|z| = \frac{1}{2}$, we have

$$(5) \qquad \left|(x^2 + z)\left(1 + \frac{y^2}{z}\right)\right| \leqq (|x|^2 + |z|)\left(1 + \left|\frac{y^2}{z}\right|\right) < \left(\frac{1}{8} + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right) < 1,$$

$$(6) \qquad \left|(x^2 + z)\frac{y^2}{z}\right| < \left(\frac{1}{8} + \frac{1}{2}\right)\cdot\frac{1}{4} < 1.$$

By (5), the equation

$$(7) \qquad z - (x^2 + z)(y^2 + z) = 0$$

has no root $z$ on the circle $|z| = \frac{1}{2}$ when $x$ and $y$ satisfy (4). The roots of (7) being continuous functions of $x$ and $y$, and reducing to 0 and 1 for $x = y = 0$, it follows that when (4) is satisfied, (7) has one root $z_1$ where $|z_1| < \frac{1}{2}$ and another $z_2$ where $|z_2| > \frac{1}{2}$. Solving (7), we find

$$(8) \qquad z_1 = \frac{1}{2}[1 - x^2 - y^2 - \sqrt{(1 + x + y)(1 + x - y)(1 - x + y)(1 - x - y)}],$$

that branch of the square root being taken which reduces to $+1$ at $x = y = 0$.

After these preliminaries, we use Cauchy's theorem on the binomial expansion of $(x^2 + z)^{m+n-2}$ to obtain

$$\frac{(m + n - 2)!}{(m - 1)!\,(n - 1)!}x^{2m-2} = \frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}(x^2 + z)^{m+n-2}\frac{dz}{z^n},$$

and consequently

$$\sum_{\substack{m+n=k\\ m\geq 1, n\geq 1}}\frac{(m + n - 2)!\,(m + n - 1)!}{m!\,(m - 1)!\,(n - 1)!\,(n - 1)!}x^{2m}y^{2n-2}$$

$$= \frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}\frac{x^2(x^2 + z)^{k-2}}{z}\sum_{n=1}^{k-1}\frac{(k - 1)!}{(k - n)!\,(n - 1)!}\frac{y^{2n-2}}{z^{n-1}}dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}\frac{x^2(x^2 + z)^{k-2}}{z}\left[\left(1 + \frac{y^2}{z}\right)^{k-1} - \left(\frac{y^2}{z}\right)^{k-1}\right]dz.$$

Since (3) is absolutely convergent by (4), it follows that

$$\frac{1}{2y}\frac{\partial F(x,y)}{\partial y}=\sum_{k=2}^{\infty}\frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}\frac{x^2(x^2+z)^{k-2}}{z}\left[\left(1+\frac{y^2}{z}\right)^{k-1}-\left(\frac{y^2}{z}\right)^{k-1}\right]dz,$$

and by (5) and (6), summation and integration may be interchanged; summing the two geometric series thus obtained under the integral sign, we find

$$\frac{1}{2y}\frac{\partial F(x,y)}{\partial y}=\frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}\frac{x^2}{z}\left[\frac{y^2+z}{z-(x^2+z)(y^2+z)}-\frac{y^2}{z-y^2(x^2+z)}\right]dz.$$

Assuming for the moment that $x \neq 0$, $y \neq 0$, it is seen that the residues of the integrand are

$$\begin{array}{ccc} 0 & \text{at} & z=0, \\ \dfrac{x^2(y^2+z_1)}{z_1(1-x^2-y^2-2z_1)} & \text{“} & z=z_1, \\ -1 & \text{“} & z=\dfrac{x^2y^2}{1-y^2}, \end{array}$$

these being the only poles inside the circle of integration. Hence, by Cauchy's theorem

$$\begin{aligned}\frac{1}{2y}\frac{\partial F(x,y)}{\partial y}&=\frac{x^2(y^2+z_1)}{z_1(1-x^2-y^2-2z_1)}-1\\&=\frac{x^2z_1-[z_1-(x^2+z_1)(y^2+z_1)]+z_1^2}{z_1(1-x^2-y^2-2z_1)}\end{aligned}$$

or, since the bracket vanishes by (7),

$$\frac{1}{2y}\frac{\partial F(x,y)}{\partial y}=\frac{x^2+z_1}{1-x^2-y^2-2z_1},$$

or finally, calculating $\partial z_1/\partial y$ from (7),

$$\frac{\partial F(x,y)}{\partial y}=\frac{\partial z_1}{\partial y}. \tag{9}$$

This, being established for $0<|x|<\frac{1}{2\sqrt{2}}$, $0<|y|<\frac{1}{2\sqrt{2}}$, also holds for $|x|+|y|<1$, both members being holomorphic in the latter region. Integrating (9) in respect to $y$, and observing that both $F(x, y)$ and $z_1$ vanish for $y = 0$, it follows that $F(x, y) = z_1$, so that (2) is proved for $|x|+|y|<1$. Now suppose that the series (1) converges for $x = x_0$, $y = y_0$, where $|x_0|>0$, $|y_0|>0$. Then, as is well known, the series converges uniformly for all $x$, $y$ satisfying the inequalities $|x| \leqq \rho|x_0|$, $|y| \leqq \rho|y_0|$, where $\rho$ is any constant less than unity; therefore $F(x, y)$ is holomorphic for all such values, in particular for $x = \rho|x_0|$, $y = \rho|y_0|$. Assuming $|x_0| + |y_0| > 1$, we may take $\rho = \dfrac{1}{|x_0| + |y_0|}$ and $F(x, y)$ would be holomorphic at a point $x, y$ where $x + y = 1$, which is impossible by (2). Hence our series diverges when $|x| + |y| > 1$, unless either $x = 0$ or $y = 0$, in which case every term vanishes and the series converges. Whether it converges or diverges for $|x| + |y| = 1$ is left undecided.